Topology and its Applications 156 (2009) 1114–1117

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Knots with $g(E(K)) = 2$ and $g(E(K \# K \# K)) = 6$ and Morimoto's Conjecture

Tsuyoshi Kobayashi [a,*], Yo'av Rieck [b]

[a] *Department of Mathematics, Nara Women's University Kitauoya Nishimachi, Nara 630-8506, Japan*
[b] *Department of Mathematical Sciences, University of Arkansas, Fayetteville, AR 72701, USA*

ARTICLE INFO

Article history:
Dedicated to the memory of Yves Mathieu and to Michel Domergue on the occasion of his 60th birthday

MSC:
57M99

Keywords:
3-Manifolds
Knots
Heegaard splittings
Tunnel number

ABSTRACT

We show that there exist knots $K \subset S^3$ with $g(E(K)) = 2$ and $g(E(K \# K \# K)) = 6$. Together with [Tsuyoshi Kobayashi, Yo'av Rieck, On the growth rate of the tunnel number of knots, J. Reine Angew. Math. 592 (2006) 63–78, Theorem 1.5], this proves existence of counterexamples to Morimoto's Conjecture [Kanji Morimoto, On the super additivity of tunnel number of knots, Math. Ann. 317 (3) (2000) 489–508]. This is a special case of [Tsuyoshi Kobayashi, Yo'av Rieck, Knot exteriors with additive Heegaard genus and Morimoto's Conjecture, Algebr. Geom. Topol. 8 (2008) 953–969, preprint version available at http://arxiv.org/abs/math.GT/0701765, 2007].

Let K_i $(i = 1, 2)$ be knots in the 3-sphere S^3, and let $K_1 \# K_2$ be their connected sum. We use the notation $t(\cdot)$, $E(\cdot)$, and $g(\cdot)$ to denote tunnel number, exterior, and Heegaard genus respectively (we follow the definitions and notations given in [7]). It is well known that the union of a tunnel system for K_1, a tunnel system for K_2, and a tunnel on a decomposing annulus for $K_1 \# K_2$ forms a tunnel system for $K_1 \# K_2$. Therefore:

$$t(K_1 \# K_2) \leqslant t(K_1) + t(K_2) + 1.$$

Since (for any knot K) $t(K) = g(E(K)) - 1$ this gives:

$$g\bigl(E(K_1 \# K_2)\bigr) \leqslant g\bigl(E(K_1)\bigr) + g\bigl(E(K_2)\bigr). \tag{1}$$

We say that a knot K in a closed orientable manifold M admits a (g, n) position if there exists a genus g Heegaard surface $\Sigma \subset M$, separating M into the handlebodies H_1 and H_2, so that $H_i \cap K$ $(i = 1, 2)$ consists of n arcs that are simultaneously parallel into ∂H_i. It is known [10, Proposition 1.3] that if K_i $(i = 1$ or $2)$ admits a $(t(K_i), 1)$ position then equality does not hold:

$$g\bigl(E(K_1 \# K_2)\bigr) < g\bigl(E(K_1)\bigr) + g\bigl(E(K_2)\bigr).$$

Morimoto proved that if K_1 and K_2 are m-small knots then the converse holds, and conjectured that this holds in general [10, Conjecture 1.5]:

* Corresponding author.
E-mail addresses: tsuyoshi@cc.nara-wu.ac.jp (T. Kobayashi), yoav@uark.edu (Y. Rieck).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2008.10.003

Conjecture 1 *(Morimoto's Conjecture). Given knots $K_1, K_2 \subset S^3$, $g(E(K_1 \# K_2)) < g(E(K_1)) + g(E(K_2))$ if and only if for $i = 1$ or $i = 2$, K_i admits a $(t(K_i), 1)$ position.*

We denote the connected sum of n copies of K by nK. We prove:

Theorem 2. *There exists infinitely many knots $K \subset S^3$ with $g(E(K)) = 2$ and $g(E(3K)) = 6$.*

Remark. This is a special case of [6, Theorem 1.3]. By specializing we obtain an easy and accessible argument that can be used as an introduction to the main ideas of [6].

As in [6] Theorem 2 implies:

Corollary 3. *There exists a counterexample to Morimoto's Conjecture, specifically, there exist knots $K_1, K_2 \subset S^3$ so that K_i does not admit a $(t(K_i), 1)$ position $(i = 1, 2)$, and (for some integer m) $g(E(K_1)) = 4$, $g(E(K_2)) = 2(m - 2)$, and $g(E(K_1 \# K_2)) < 2m$.*

Proof. This argument originally appeared in the proof of [5, Theorem 1.5]. For details see the proof of [6, Corollary 1.6]. □

We note that K_1 and K_2 are composite knots. This leads Moriah [9, Conjecture 7.14] to conjecture that if K_1 and K_2 are prime then Conjecture 1 holds.

1. The proof

Let X be the exterior of a knot K in a closed orientable manifold. For an integer $c \geqslant 0$ let $X^{(c)}$ denote the manifold obtained by drilling c curves out of X that are simultaneously parallel to meridians of K. The following is [6, Proposition 2.2], where the proof can be found. Note the relation to [13, Theorem 3.8].

Proposition 4. *Let X, $X^{(c)}$ be as above and $g \geqslant 0$ an integer. Suppose $X^{(c)}$ admits a strongly irreducible Heegaard surface of genus g. Then one of the following holds*:

(1) *X admits an essential surface S with $\chi(S) \geqslant 4 - 2g$.*
(2) *For some b, $c \leqslant b \leqslant g$, K admits $(g - b, b)$ position.*

Given an integer $d > 0$, Johnson and Thompson [4] and Minsky, Moriah and Schleimer [8] construct infinitely many knots $K \subset S^3$ so that $E(K)$ admits a genus 2 Heegaard splitting of distance more than d (in the sense of the curve complex [2]). (Note that [8] is more general.) Fix such a knot K for $d = 10$. The two properties of K we will need are described in the lemmas below:

Lemma 5. *X does not admit an essential surface S with $\chi(S) \geqslant -8$.*

Proof. This follows directly from [12, Theorem 31.]. □

Lemma 6. *K does not admit a $(0, 3)$ or a $(1, 2)$ position.*

Proof. Assume, for a contradiction, K admits a $(0, 3)$ or a $(1, 2)$ position. By [4, Theorem 1], if K admits a (p, q) position (for some p, q) then either K is isotopic into a genus p Heegaard surface, or the distance of any Heegaard splitting of X is at most $2(p + q)$. Since K is not a trivial knot or a torus knot, the former cannot happen. (Note that, by [15] we see that the distance of each Heegaard splitting of the exterior of any torus knot is at most 2.) On the other hand, if the latter holds, then the distance of any Heegaard splitting of X should be at most 6 contradicting our choice of K. □

For integers $n \geqslant 1$ and $c \geqslant 0$ we denote the exterior of nK by $X(n)$, and the manifold obtained by drilling c curves out of $X(n)$ that are simultaneously parallel to meridians of nK by $X(n)^{(c)}$.

Thus we obtain $X(n)^{(c)}$ by drilling a curve $\gamma_n \subset X(n)^{(c-1)}$ that is parallel to ∂X, and in particular, γ_n can be isotoped onto any Heegaard surface of $X^{(c-1)}$. This is described in [11] by saying that $X^{(c-1)}$ is obtained from $X^{(c)}$ by a *good* Dehn filling. For good Dehn fillings [11] shows (see the proof of Theorem 5.1 of [7] for details):

Lemma 7. *Either $g(X(n)^{(c)}) = g(X(n)^{(c-1)})$ or $g(X(n)^{(c)} = g(X(n)^{(c-1)}) + 1$.*

Lemma 8. $g(X^{(1)}) = 3$.

Proof. Since $g(X)=2$, by Lemma 7 $g(X^{(1)})=2$ or $g(X^{(1)})=3$. Assume for a contradiction that $g(X^{(1)})=2$ and let $\Sigma^{(1)} \subset X^{(1)}$ be a minimal genus Heegaard surface.

Claim. *$\Sigma^{(1)}$ is strongly irreducible.*

Proof. Suppose $\Sigma^{(1)}$ weakly reduces. Then by Casson and Gordon [1] (see [16, Theorem 1.1] for a relative version) an appropriately chosen weak reduction yields an essential surface S with $\chi(S) \geqslant \chi(\Sigma^{(1)})+4=2$. Since $X^{(1)}$ does not admit an essential sphere this is a contradiction, proving the claim. $\square$

Thus we may assume $\Sigma^{(1)}$ is strongly irreducible. By Proposition 4, either X admits an essential surface S with $\chi(S) \geqslant 4-2g(\Sigma^{(1)})=0$ or K admits a $(2-b,b)$ position, with $1 \leqslant b \leqslant 2$. The former contradicts Lemma 5. For the latter, we get a $(1,1)$ position (for $b=1$) or a $(0,2)$ position (for $b=2$). Both contradict Lemma 6. $\square$

Lemma 9. $g(X^{(2)})=4$.

Proof. Since $g(X^{(1)})=3$, by Lemma 7 $g(X^{(2)})=3$ or $g(X^{(2)})=4$. Assume for a contradiction that $g(X^{(2)})=3$ and let $\Sigma^{(2)} \subset X^{(2)}$ be a minimal genus Heegaard surface.

Claim. *$\Sigma^{(2)}$ is strongly irreducible.*

Proof. Suppose $\Sigma^{(2)}$ weakly reduces. Then by Casson and Gordon [1] (see [16] for a relative version) an appropriately chosen weak reduction yields an essential surface S with $\chi(S) \geqslant \chi(\Sigma^{(2)})+4=0$. Since $X^{(2)}$ does not admit an essential sphere, this surface must be a collection of tori; let F be one of these tori. By [7, Proposition 2.13], $\Sigma^{(2)}$ weakly reduces to F.

Note that $X^{(2)}$ admits an essential torus T giving the decomposition $X^{(2)}=X' \cup_T Q^{(2)}$, where $Q^{(2)}$ is homeomorphic to an annulus with two holes cross S^1 and $X' \cong X$.

Since F and T are incompressible, we may suppose that each component of $F \cap T$ is a simple closed curve which is essential in both F and T. Minimize $|F \cap T|$ under this constraint. We claim that $F \cap T=\emptyset$. Assume for a contradiction $F \cap T \neq \emptyset$. Then any component of $F \cap X'$ is an essential annulus; by Lemma 5, X' does not admit essential annuli.

Thus we may assume $F \subset X'$ or $F \subset Q^{(2)}$. If $F \subset X'$ and not parallel to T then $X \cong X'$ is toroidal, contradicting Lemma 5. If F is parallel to T we isotope it into $Q^{(2)}$.

Thus we may assume $F \subset Q^{(2)}$. By [3, VI.34] F is a vertical torus in $Q^{(2)}$. Assume first that F is isotopic to a component of $\partial Q^{(2)}$. Since F was obtained by weakly reducing a minimal genus Heegaard surface for $X^{(2)}$, by [16, Theorem 1.1] F is not peripheral, *i.e.*, F is not isotopic to a component of $\partial X^{(2)}$. Hence F is isotopic to T and $X^{(2)}=X' \cup_F Q^{(2)}$. Note that by [14] $g(Q^{(2)})=3$, and since $X \cong X'$, $g(X')=2$. Since F was obtained by weakly reducing a *minimal genus* Heegaard surface [7, Proposition 2.9] (see also [14, Remark 2.7]) gives:

$$g\big(X^{(2)}\big)=g\big(Q^{(2)}\big)+g(X')-g(F)=3+2-1=4.$$

This contradicts our assumption that $g(X^{(2)})=3$.

Next assume that F is not isotopic to a component of $\partial Q^{(2)}$. Then F is isotopic to a vertical torus giving the decomposition $X^{(2)}=X_1 \cup_F D(2)$, where X_1 is homeomorphic to $X^{(1)}$ and $D(2)$ is homeomorphic to a twice punctured disk cross S^1. By Lemma 8 $g(X_1)=3$ and by [14] $g(D(2))=2$. We get:

$$g\big(X^{(2)}\big)=g(X_1)+g\big(D(2)\big)-g(F)=3+2-1=4.$$

This contradicts our assumption that $g(X^{(2)})=3$.

This contradiction proves the claim. $\square$

Thus we may assume $\Sigma^{(2)}$ is strongly irreducible. By Proposition 4, either X admits an essential surface S with $\chi(S) \geqslant 4-2g(\Sigma^{(2)})=-2$ or K admits $(g(\Sigma^{(2)})-b,b)=(3-b,b)$ position, with $2 \leqslant b \leqslant 3$. The former contradict Lemma 5. For the latter, we get a $(1,2)$ position (for $b=2$) or a $(0,3)$ position (for $b=3$). Both contradict Lemma 6. $\square$

Lemma 10. $g(X(2))=4$.

Proof. By inequality (1) $g(X(2)) \leqslant 4$. Therefore by the Swallow Follow Torus Theorem [7, Theorem 4.1] and Lemma 5 any minimal genus Heegaard surface for $X(2)$ weakly reduces to a swallow follow torus F, giving the decomposition: $X(2)=X^{(1)} \cup_F X$. By [7, Proposition 2.9] and Lemma 8, $g(X(2))=g(X^{(1)})+g(X)-g(F)=3+2-1=4$. $\square$

Lemma 11. $g(X(2)^{(1)})=5$.

Proof. By Lemmas 7 and 10, $g(X(2)^{(1)}) = 4$ or $g(X(2)^{(1)}) = 5$. Assume for a contradiction that $g(X(2)^{(1)}) = 4$. By the Swallow Torus Theorem [7, Theorem 4.2] and Lemma 5 any minimal genus Heegaard surface for $X(2)^{(1)}$ weakly reduces to a swallow follow torus F giving one of the following decompositions:

(1) $X(2)^{(1)} = X(2) \cup_F Q^{(1)}$, where $Q^{(1)}$ is homeomorphic to an annulus with one hole cross S^1.
(2) $X(2)^{(1)} = X^{(1)} \cup_F X^{(1)}$.
(3) $X(2)^{(1)} = X^{(2)} \cup_F X$.

By [14] $g(Q^{(1)}) = 2$; the genera of all other manifolds are given in the lemmas above. By amalgamation [7, Proposition 2.9] we get:

(1) $g(X(2)^{(1)}) = g(X(2)) + g(Q^{(1)}) - g(F) = 4 + 2 - 1 = 5$.
(2) $g(X(2)^{(1)}) = g(X^{(1)}) + g(X^{(1)}) - g(F) = 3 + 3 - 1 = 5$.
(3) $g(X(2)^{(1)}) = g(X^{(2)}) + g(X) - g(F) = 4 + 2 - 1 = 5$. □

Proof of Theorem 2. By inequality (1), $g(X(3)) \leqslant 6$. Therefore, by the Swallow Follow Torus Theorem [7, Theorem 4.2] and Lemma 5 any minimal genus Heegaard surface for $X(3)$ weakly reduces to a swallow follow torus F giving one of the following decompositions:

(1) $X(3) = X^{(1)} \cup_F X(2)$.
(2) $X(3) = X(2)^{(1)} \cup_F X$.

The genera of the manifolds are given in the lemmas above. By amalgamation [7, Proposition 2.9] we get:

(1) $g(X(3)) = g(X^{(1)}) + g(X(2)) - g(F) = 3 + 4 - 1 = 6$.
(2) $g(X(3)) = g(X(2)^{(1)}) + g(X) - g(F) = 5 + 2 - 1 = 6$.

This completes the proof of Theorem 2. □

References

[1] A.J. Casson, C.McA. Gordon, Reducing Heegaard splittings, Topology Appl. 27 (3) (1987) 275–283.
[2] John Hempel, 3-manifolds as viewed from the curve complex, Topology 40 (3) (2001) 631–657.
[3] William Jaco, Lectures on Three-Manifold Topology, CBMS Reg. Conf. Ser. Math., vol. 43, American Mathematical Society, Providence, RI, 1980.
[4] Jesse Johnson, Abigail Thompson, On tunnel number one knots which are not $(1,n)$, available at http://arxiv.org/abs/math.GT/0606226, 2006.
[5] Tsuyoshi Kobayashi, Yo'av Rieck, On the growth rate of the tunnel number of knots, J. Reine Angew. Math. 592 (2006) 63–78.
[6] Tsuyoshi Kobayashi, Yo'av Rieck, Knot exteriors with additive Heegaard genus and Morimoto's Conjecture, Algebr. Geom. Topol. 8 (2008) 953–969, preprint version available at http://arxiv.org/abs/math.GT/0701765, 2007.
[7] Tsuyoshi Kobayashi, Yo'av Rieck, Heegaard genus of the connected sum of m-small knots, Comm. Anal. Geom. 14 (5) (2006) 1037–1077, preprint version available at http://arxiv.org/abs/math/0503229.
[8] Yair Minsky, Yoav Moriah, Saul Schleimer, High distance knots, Algebr. Geom. Topol. 7 (2007) 1471–1483, preprint version available at http://arxiv.org/abs/math.GT/0607265, 2006.
[9] Yoav Moriah, Heegaard splittings of knot exteriors, Geom. Topol. Monogr. 12 (2007) 191–232, preprint version available at http://arxiv.org/abs/math.GT/0608137, 2006.
[10] Kanji Morimoto, On the super additivity of tunnel number of knots, Math. Ann. 317 (3) (2000) 489–508.
[11] Yo'av Rieck, Heegaard structures of manifolds in the Dehn filling space, Topology 39 (3) (2000) 619–641.
[12] Martin Scharlemann, Proximity in the curve complex: Boundary reduction and bicompressible surfaces, Pacific J. Math. 228 (2) (2006) 325–348, preprint version available at http://arxiv.org/abs/math.GT/0410278, 2004.
[13] Martin Scharlemann, Jennifer Schultens, Comparing Heegaard and JSJ structures of orientable 3-manifolds, Trans. Amer. Math. Soc. 353 (2) (2001) 557–584 (electronic).
[14] Jennifer Schultens, The classification of Heegaard splittings for (compact orientable surface) $\times S^1$, Proc. London Math. Soc. (3) 67 (2) (1993) 425–448.
[15] Jennifer Schultens, Heegaard splittings of Seifert fibered spaces with boundary, Trans. Amer. Math. Soc. 347 (1995) 2533–2552.
[16] Eric Sedgwick, Genus two 3-manifolds are built from handle number one pieces, Algebr. Geom. Topol. 1 (2001) 763–790 (electronic).